# A METHOD FOR INCORPORATING IMAGE FORCES IN MULTIPARTICLE TRACKING WITH SPACE CHARGE

F.W. Jones, TRIUMF, 4004 Wesbrook Mall, Vancouver, V6T 2A3, Canada
H.O. Schönauer, CERN, CH-1211 Geneva 23, Switzerland

*Abstract*

The simulation of intense beams in synchrotrons or storage rings requires the rapid evaluation of space-charge forces for large macroparticle ensembles. This usually dictates the use of the most economical mesh-based field solvers, such as those based on Fourier-transform methods, which provide essentially a free-space solution for the field in the beam region and do not include image effects due to the vacuum chamber. We describe here a method, recently implemented in the code Accsim[1], for accurately deriving the image-force contributions for arbitrary beam distributions and vacuum chamber geometries, with only a modest increase in computing time. Accsim's Hybrid-Fast-Multipole space charge model[2] lends itself particularly well to this approach, which uses harmonic functions and a minimization technique to obtain a parameterized solution for the image potential. The details of the procedure are presented, along with evaluations of its speed and accuracy and some results of validation tracking tests.

## 1 INTRODUCTION

Most generally, the self-consistent tracking of a simulated beam with space-charge entails the stepwise integration of particle motion via repeated "field-solve" and "particle-push" operations, where the space-charge field due to the macroparticle ensemble must be calculated and each particle advanced by a suitably small distance or time interval with appropriate deflections due to the space-charge and external focussing forces.

Although many variants of this scheme have been devised in order to achieve better accuracy, economy, or numerical stability, the requirement for rapid field-solve operations remains paramount even in linear structures and is particularly severe in rings where many turns must be simulated, as in current studies of proton drivers for spallation sources and neutrino factories.

For these applications the class of 2D mesh-based "Rapid Elliptic Solvers"[3], using cyclic reduction, FFT methods, etc., offer the best performance but they do not allow arbitrary boundary shapes and are limited to simple boundary conditions at the edges of the problem space. Moreover, extending a mesh from the beam region to enclose the vacuum chamber will quadratically scale up the number of mesh points and hence the computation time.

If, as is often the case, the image forces due to the vacuum chamber can be neglected, then it is most economical to perform a "free-space" solution using one of the fast Fourier-based solvers, where the boundary conditions are implicitly periodic but the system of charges can be isolated from its false images by using a solution mesh that extends somewhat beyond the edges of the beam. This approach has been used profitably in contemporary large-scale spallation-source studies.[4]

For the code Accsim, often used to study injection and accumulation of intense beams, there are some existing and potential applications where image forces may have significant impact on machine performance. In the following sections we outline our implementation of a new method for including these forces, by adding to the direct (self-field) calculation a second stage which efficiently and accurately solves for the image field. Here we take particular advantage of the stored self-field data that is a by-product of Accsim's Hybrid-Fast-Multipole solver. However, this image-field method can be applied in any scenario in which it is possible to derive sufficiently accurate estimates for the direct field at arbitrary locations outside the beam and possibly outside the solution mesh as well.

## 2 PROBLEM DEFINITION

The Hybrid Fast-Multipole routines in Accsim provide a solution $V(x, y)$ for the space-charge potential due to the beam in free space. This is a solution to Poisson's equation without any imposed boundary conditions.

Suppose $U(x, y)$ is a solution to Laplace's equation such that $U = -V$ on the boundary defined by the vacuum chamber wall. Then $W = V + U$ is a solution to Poisson's equation satisfying the boundary condition $W = 0$ (or $E_{\parallel} = 0$) at the vacuum chamber wall.

Thus $U$ represents the image potential of the beam in the vacuum chamber. In the following, we outline a method to obtain a parameterized solution for $U$ and its partial derivatives, whence the image force on each particle can be computed and applied, in like manner to the self-field force, during each integration step of the space-charge tracking.

## 3 EXPANSION OF IMAGE POTENTIAL

We expand $U$ to order $N$ in harmonic functions,

$$U(r, \theta) = \sum_{n=0}^{N} [a_n r^n \cos n\theta + b_n r^n \sin n\theta],$$

where ($r$,$\theta$) are the usual polar coordinates and $r$ is normalized to $[0, 1]$ with respect to a circle sufficiently large to enclose the vacuum chamber. We would like to find coefficients $a_n, b_n$ which minimize

$$I = \int_0^{2\pi} [V(r,\theta) - U(r,\theta)]^2 d\theta,$$

or, separating variables and differentiating:

$$2\int_0^{2\pi} r^n \cos n\theta \, [V-U] \, d\theta = 0 \qquad n = 0, \ldots N$$

$$2\int_0^{2\pi} r^n \sin n\theta \, [V-U] \, d\theta = 0 \qquad n = 0, \ldots N$$

If we introduce some shorthand

$$\psi_n = r^n \cos n\theta, \quad \phi_n = r^n \sin n\theta$$

$$A_{ik} = \int_0^{2\pi} \phi_i \phi_k d\theta, \quad B_{ik} = \int_0^{2\pi} \psi_i \psi_k d\theta$$

$$C_{ik} = \int_0^{2\pi} \psi_i \phi_k d\theta$$

$$\Phi_{c,n} = \int_0^{2\pi} V \psi_n d\theta, \quad \Phi_{s,n} = \int_0^{2\pi} V \phi_n d\theta,$$

then the system of equations reduces to:

$$\Phi_{c,n} - \sum_{k=0}^{N} a_k B_{nk} - \sum_{k=0}^{N} b_k C_{nk} = 0$$

$$\Phi_{s,n} - \sum_{k=0}^{N} a_k C_{kn} - \sum_{k=0}^{N} b_k A_{kn} = 0$$

## 4 SOLUTION FOR IMAGE POTENTIAL

We now invoke the four-fold mirror symmetry of the vacuum chamber to make the coupled terms $C_{ik}$ vanish, leaving us with

$$\Phi_{c,n} - \sum_{k=0}^{N} a_k B_{nk} = 0,$$

$$\Phi_{s,n} - \sum_{k=0}^{N} b_k A_{kn} = 0.$$

The integrals $A_{kn}, B_{nk}$ can be estimated numerically by summing over a sufficient number (minimum $2N+1$) of boundary points $(r_b, \theta_b)$ around the vacuum chamber.

For the integrals $\Phi_{c,n}, \Phi_{s,n}$ we must additionally compute $V(r_j, \theta_j)$ at each boundary point. In Accsim this is a simple matter because the fast-multipole solver, invoked beforehand to find the direct field, provides a re-entry routine to evaluate the multipole expansions which it has retained in static storage.

Once the integrals have been computed, we have two linear systems which can be solved quite economically by matrix inversion, yielding the expansion coefficients $a_k, b_k$.

The above procedures were implemented in a new Accsim routine TSIMCO which, given the set of boundary coordinates and the order to which the image-potential expansion should be taken, computes the boundary integrals and expansion coefficients, utilizing the CERN Library routine RINV for matrix inversion.

## 5 VACUUM CHAMBER GEOMETRIES

To determine how faithfully various chamber geometries could be modelled by this method, tests were conducted for circular, elliptical and rectangular chambers containing a beam of uniform elliptical cross section, represented by $10^4$ macroparticles. In each case, the beam was placed off-center in the chamber and fairly close to the wall (clearance of about one beam half-height between beam edge and wall) in order to obtain an image field that was comparable in strength to the self-field. Some results, indicating for different expansion orders how nearly the chamber wall could be brought to zero potential, are shown in Table 1 and graphically in Figure 1.

| Vacuum chamber | Expansion order | Boundary points | Maximum error (%) |
|---|---|---|---|
| Circular | 2 | 8 | 1.00 |
| Circular | 3 | 8 | 0.16 |
| Elliptical | 5 | 18 | 4.38 |
| Elliptical | 7 | 18 | 1.30 |
| Rectangular | 9 | 32 | 2.92 |
| Rectangular | 11 | 32 | 1.31 |

Table 1: Conformance of image-field solution to vacuum chamber shape, measured as $\max|(V+U)/V|$, for an off-center uniform elliptical beam.

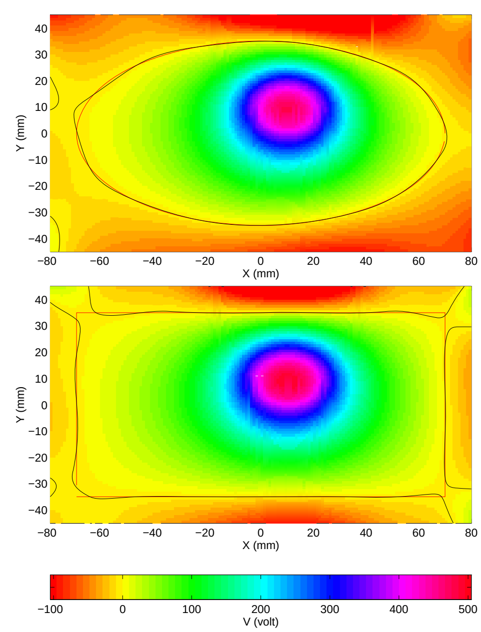


Figure 1: Computed space-charge potential (self+image) of an offset uniform beam in elliptical and rectangular chambers. Red line indicates chamber wall and black lines are zero-potential contours.

In further testing we found that the solution accuracy, which is strongly dependent on the image-potential expansion order $N$, is also weakly dependent on the number of boundary points used to represent the chamber, and it is usually profitable to increase this number somewhat above the minimum of $2N+1$.

As seen in Figure 1, the harmonic-function representation of the vacuum chamber can be considered as a somewhat distorted version of the real chamber and the image expansion order can be chosen by considering what is an acceptable level of distortion for given beam and chamber dimensions.

## 6 COMPUTATION TIMES

By detailed timing of the calculations for self-field, image-field and tracking-step, we confirmed that the computational cost of adding image forces to the tracking is not excessive. For example, the following timings on a DEC Alpha 333MHz processor were obtained for a beam of uniform cross section with $10^4$ macroparticles, a direct-field solution grid of 161×161, and an elliptical vacuum chamber represented by 64 boundary points. A 7th-order expansion was used for the image potential.

| | |
|---|---|
| Self: grid assignment | 0.018 sec |
| Self: field-solve (FMM) | 0.132 sec |
| Image: boundary evaluation | 0.001 sec |
| Image: solve for coefficients | 0.013 sec |
| Self: compute per-particle force | 0.009 sec |
| Image: compute per-particle force | 0.044 sec |

In this and other test cases we found that the image-field solution time is quite small compared to the self-field solution, and that the more significant cpu time increase is in the computation of the image-field expansions, which must be evaluated at each particle location to obtain the force components acting on the particle. However, even for eccentric chambers requiring higher-order expansions, the time increase needed to add image forces to the tracking is quite reasonable and both complements and benefits from the efficiency of the the hybrid-fast-multipole solver.

## 7 COHERENT IMAGE TUNE SHIFT

A sensitive test of tracking with image forces is to observe a beam with a small coherent oscillation in a low-energy regime. The coherent tune should be unaffected by the direct space-charge field but should respond to the image field according to:[5]

$$\Delta Q_x^{\rm coh} = -\frac{N r_0}{\pi\beta^2\gamma}\bar{\beta}_x\,\frac{\xi_{1,x}}{h^2}$$

(and similarly for $y$), where $\bar{\beta}_x$ is the lattice beta function averaged over the ring circumference and $h$ is the vacuum chamber half-height. For a circular chamber, the image coefficients are $\xi_{1,x}{=}\xi_{1,y}{=}1/2$.

For different beam intensities we performed coherent tune measurements (by FFT of the beam centroid sampled after each lattice cell) for a coasting KV beam in a realistic lattice (CERN PS Booster) but with a small ($h = 30$mm) circular vacuum chamber to intensify the image effect. The beam was initially given a small offset of (2mm,2mm) from the center of the chamber and then tracked for 50 turns.

The bare tunes for the lattice are $(Q_x, Q_y){=}(4.28, 5.55)$. At 50 MeV beam energy, emittances of $\varepsilon_x = \varepsilon_y = 45$ mm-mr, and an intensity of $N \sim 10^{13}$, the self-field forces are quite large, with incoherent tune shifts of around 0.4–0.5, but the small chamber size also yields significant image-field coherent tune shifts of around 0.15.

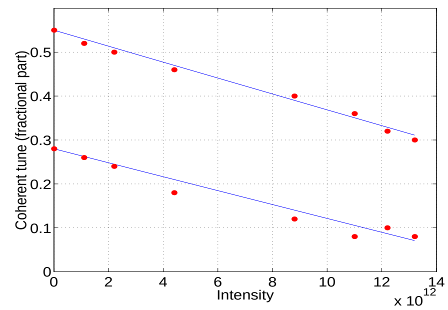

Figure 2: Coherent image tune shifts in PSB lattice with artificially small vacuum chamber. Lines represent predictions of tune-shift formula and markers indicate results from tracking data.

Figure 2 shows the results of coherent tune measurements at various intensities, as compared with the predictions of the above tune-shift formula. We found that good agreement with the predicted tune shifts could be maintained over a wide range of intensities. Raising the intensity to $N{\simeq}1.75{\cdot}10^{13}$ brought the beam onto the integer resonance $Q_x{=}4$, where we observed unstable motion and particle loss on horizontal apertures. These results indicate that although the image field is parameterized by fitting potentials at the vacuum chamber radius, this yields accurate field values even near the center of the chamber.

## 8 CONCLUSION

An economical method of including image forces in space-charge tracking has been developed and will be made available in future releases of the simulation code Accsim.

## 9 REFERENCES


[1] F.W. Jones and H. Schönauer, "New Space-Charge Methods in Accsim and Their Application to Injection in the CERN PS Booster," Proc. 1999 Particle Accelerator Conf., New York, March 1999.

[2] F.W. Jones, "A Hybrid Fast-Multipole Technique for Space-Charge Tracking With Halos," *Workshop on Space Charge Physics in High Intensity Hadron Rings*, Shelter Island NY, May 1998, AIP Conf. Proc. **448**, p. 359.

[3] R.W. Hockney and J.W. Eastwood, *Computer Simulation Using Particles*, Bristol and New York: Adam Hilger, 1988.

[4] J.A. Holmes et al., "Space charge dynamics in high intensity rings," Phys. Rev. Special Topics **2**: 114202 (1999).

[5] P. Bryant and K. Johnsen, *The Principles of Circular Accelerators and Storage Rings*, Cambridge University Press, 1993.