

THE NUCLEAR SHELL MODEL WITH APPLICATIONS TO ASTROPHYSICS

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Lecture 1: **One at a time: the non-interacting shell model**

Lecture 2: **Mixing it up: the interacting shell model**

Lecture 3: **Talking to leptons: weak and EM transitions**

Lecture 4: **Cooking in the cosmic kitchen:
applications to astrophysics**

Lecture 1: ONE AT A TIME: THE NON-INTERACTING SHELL MODEL

Goal: **microscopic** model of nuclei =
quantum wavefunction with protons, neutrons as
degrees of freedom

Schrödinger eqn: $\hat{H}\Psi = E\Psi$

For A particles $\Psi = \Psi(x_1, x_2, x_3, \dots, x_A)$

$$\hat{H} = \sum_{i=1}^A -\frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} \sum_{i \neq j} V(x_i, x_j) + \sum_{i=1}^A V_{central}(x_i)$$

for atoms,
not for nuclei

A differential equation in $3A$ dimensions!

Cannot be solved even numerically for $A > 10$.

“EXACT” SOLUTIONS OF NUCLEAR STRUCTURE

Realistic, high-precision interactions from NN scattering:

Nijmegen

Bonn

Argonne

Reid 93

Techniques for full numerical solutions:

Faddeev (for $A = 3$ or 4)

hyperspherical harmonics (for $A=3$ or 4)

variational + Green's function (diffusion) Monte Carlo ($A=3-8+$)

no-core shell model + effective interaction ($A=3-8+$)

THE ONLY THING WE REALLY KNOW HOW TO SOLVE:

Schrödinger equation for one particle (in one dimension!)

Also known as the **independent particle** model.

$$\Psi(x_1, x_2, \dots, x_A) = \phi_1(x_1)\phi_2(x_2)\dots\phi_A(x_A)$$

But fermionic wavefunctions require antisymmetry:

$$\Psi(\dots, x_a, \dots, x_b, \dots) = -\Psi(\dots, x_b, \dots, x_a, \dots)$$

Introduce “Slater
determinant”:

$$\Psi = \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_A) \\ \phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_A(x_1) & \phi_A(x_2) & \cdots & \phi_A(x_A) \end{vmatrix}$$

The Slater determinant has all the necessary properties: antisymmetry, orthonormality, etc. For example, if $x_1=x_2$, or $\phi_1=\phi_2$, then the determinant vanishes

The ϕ_i are the
single-particle states

Without loss of
generality assume
single-particle states
are orthonormal:

$$\int dx \phi_a^*(x)\phi_b(x) = \delta_{ab}$$

VARIATIONAL PRINCIPLE → HARTREE-FOCK EQUATION

With the independent particle wavefunction in hand, we want to change the **many-body** Schrödinger eqn to an approximate **one-body** equation.

We do this by invoking the variational principle

$$\text{minimize } E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

This is not as hard as it seems, because the single-particle states are **uncorrelated** and **orthonormal**

The Hartree-Fock equation: N differential equations, each for **one** particle wavefunction!

$$-\frac{\hbar^2}{2m} \nabla^2 \phi_a(x) + V_{Hartree}(x) \phi_a(x) - \int dy V_{ex}(x, y) \phi_a(y) = \epsilon_a \phi_a(x)$$

“single-particle energy”

oops! nonlocal! bad news!

$$V_{Hartree}(x) = \sum_a \int dy V(x, y) \phi_b(y) \phi_b(x) \quad V_{ex}(x, y) = \sum_a \int dy V(x, y) \phi_b(y) \phi_b(x)$$

TROUBLE SPOTS FOR HARTREE-FOCK

1. Repulsive short-range interactions (“repulsive core”)

HF, which is low-energy, long-distance approximation, does not handle well the known “repulsive core” of NN interaction.

Solutions: (1) use phenomenological interactions w/o repulsive core
(2) “integrate out” short-range repulsion from realistic interaction

2. Exchange forces these are typically long-range, nonlocal, and very complicated.

Solutions: (1) ignore exchange = “Hartree approximation”

(2) treat exchange by local, phenomenological approximation (see atomic)
(3) use phenomenological force for which exchange is easy, such as delta-force (Skyrme)

3. Saturation: It is difficult for a purely two-body interaction to reproduce simultaneously both binding energies and radii

solutions (1) density-dependent interactions (2) relativity

NN INTERACTIONS FOR HF

(1) Skyrme *NN* interaction approximated by $\delta(r_1-r_2)$.
Easy to apply; analogous to *local density approximation*;
exchange force simpler; must include density-dependence to reproduce data

(2) Relativistic *NN* interaction as sum of Lorentz scalar, vector.
Reproduces data (because in non-relativistic reduction, yields an effective density-dependent force). Solve Dirac-Hartree (Fock term too difficult).

(3) Brueckner *G*-matrix start from **realistic** *NN* interaction and integrate out short-range degrees of freedom. Most consistent, but very time-consuming. Effective interaction also has induced density-dependent forces.

COMPARISON WITH ATOMIC/MOLECULAR

$V(1,2) = \text{Coulomb...}$ should be easy, right?

Direct or Hartree term is easy:
compute charge density ρ ,
solve Poisson eqn $\nabla^2\Phi=4\pi\rho$

But exchange term is a nightmare:
long-range, nonlocal, unworkable

Instead: use phenomenological interaction!

use a **local**, density-dependent term to mock up exchange term,
+ correlation energy left out by mean-field theory.

e.g., $V_{\text{exc}} \propto \rho^{-2/3}$

These can be “derived” by assuming an infinite, homogeneous electron gas, or through Monte Carlo simulations

This is called “Density functional” or Kohn-Sham theory
and in the end is not that much different from what nuclear physicists do!

HARTREE-FOCK FOR SPHERICAL NUCLEI

The HF eqn is for **one** particle but in **three** dimensions

$$-\frac{\hbar^2}{2m} \nabla^2 \phi_a(\vec{r}) + U_{HF}(\vec{r}) \phi_a(\vec{r}) = \epsilon_a \phi_a(\vec{r})$$

↖ the exchange (Fock) potential can be nonlocal..

If we assume spherical symmetry, we can use **separation of variables** for the single-particle wavefunction and get a **radial HF eqn in one-d!**

$$\phi_{nlm}(\vec{r}) = \frac{R_{nl}(r)}{r} Y_{lm}(\Omega)$$

↖

n = nodal or radial quantum number

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} R_{nl}(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} R_{nl}(r) + U_{HF}(r) R_{nl}(r) = \epsilon_{nl} R_{nl}(r)$$

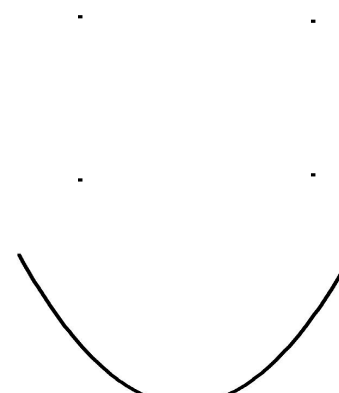
Much easier to solve!

APPROXIMATE SINGLE-PARTICLE POTENTIALS

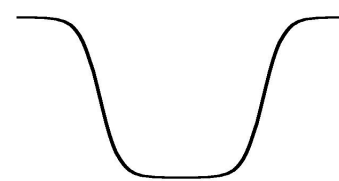
Even radial Hartree-Fock is difficult and time-consuming,, especially since no **single** HF interaction. →

Often use a simpler, analytic potential that shares same features as Hartree-Fock

Spherical well U =



Harmonic oscillator U =



$-V_0 \frac{1}{1 + e^{(r-R)/a}}$ Woods-Saxon U =

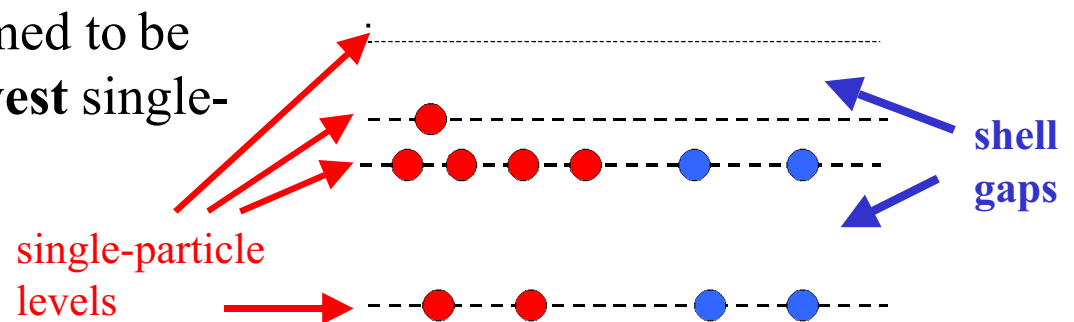
MAGIC NUMBERS

When we solve the HF eqn, we get out: single-particle wfns ϕ
single-particle energies ϵ

If the HF potential has rotational symmetry, then some single-particle energies will be degenerate.

We can put only **one fermion** into each single-particle orbit, but several fermions may have degenerate single-particle energies (as long as some quantum number, such as M or S_z is different).

The **Hartree-Fock state** is assumed to be a Slater determinant with the **lowest** single-particle energies occupied.



A large energy difference between single-particle levels is called a **shell gap** and the number of particles it takes to fill up to a shell gap is called a **magic number** -- like noble gases

THE IMPORTANCE OF SPIN-ORBIT SPLITTING

Nuclear magic numbers = 2, 8, 20, 50, 82. . .

harmonic oscillator magic #s = 2, 8, 20, 40, 70

However, no simple potential (square well, harmonic oscillator, Woods-Saxon) **alone** can explain these magic numbers (also known as shell closures)....

...that is, unless one includes spin-orbit splitting, which means the radial Schrodinger eqn depends not only on l but also on j of the single-particle state

Spin-orbit splitting occurs in atoms but is very small. When first proposed, a large spin-orbit force for nuclei seemed radical.

Now we know it is a very **natural** consequence. Spin-orbit forces arise from relativistic effects, and nucleons bound in a nucleus are more relativistic than electrons in atoms.

SINGLE-PARTICLE LEVELS

orbital ang. mom. $l \dots s=0, p=1, d=2, f=3, g=4 \dots$ (parity $\pi=(-1)^l$)

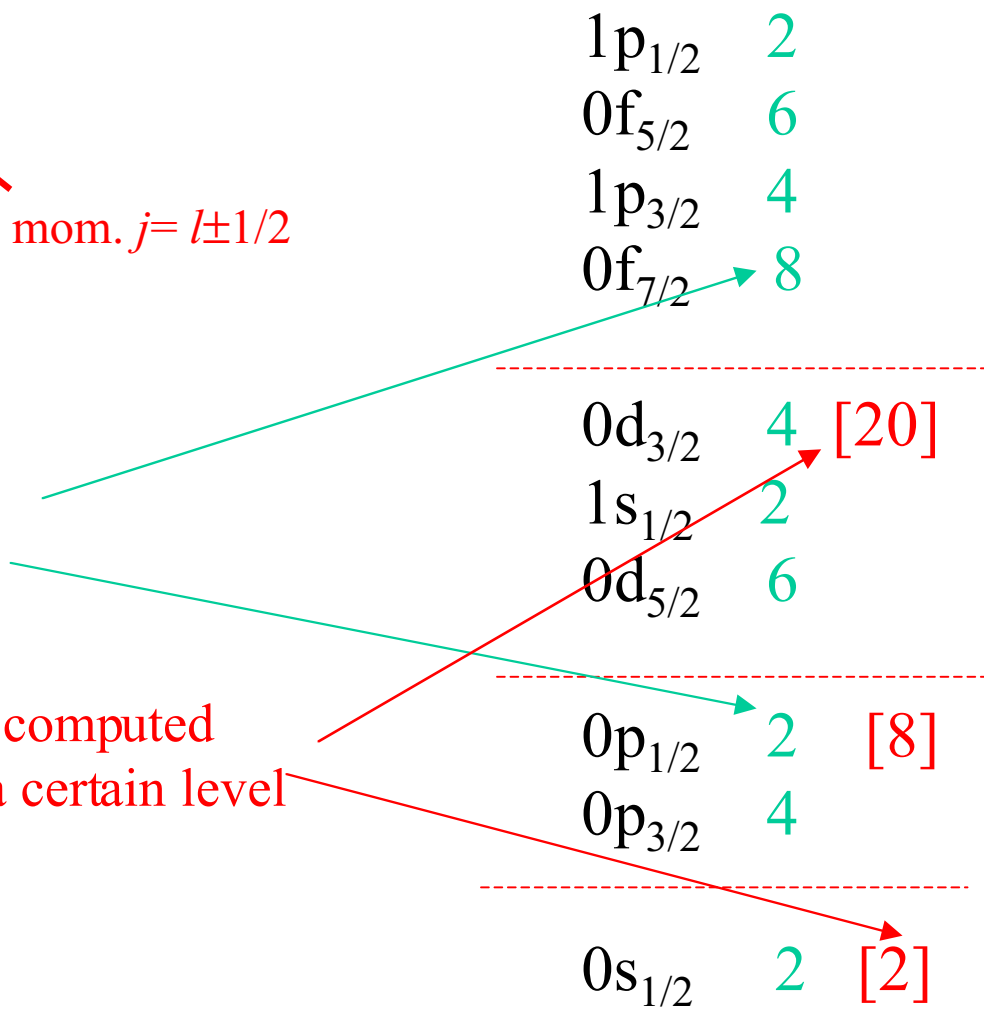
spectroscopic notation: $0 f_{7/2}$

“nodal” quantum number 0 total ang. mom. $j = l \pm 1/2$

For a given species of nucleon
(protons *or* neutrons) we can put
 $2j+1$ particles into a j -orbit.

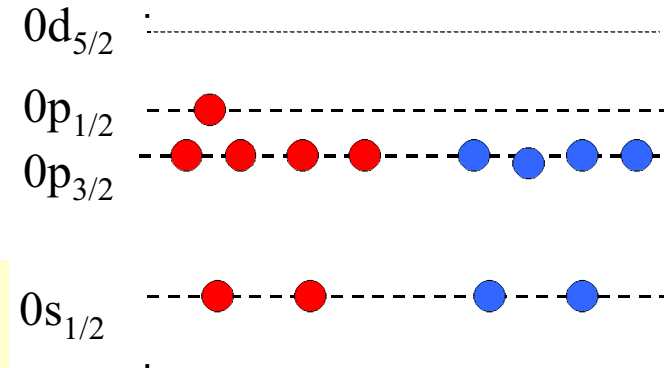
Shell gaps or **magic numbers** can be computed
by adding up the occupancies below a certain level

$1d_{5/2}$		
$0g_{9/2}$	10	[50]
$1p_{1/2}$	2	
$0f_{5/2}$	6	
$1p_{3/2}$	4	
$0f_{7/2}$	8	
$0d_{3/2}$	4	[20]
$1s_{1/2}$	2	
$0d_{5/2}$	6	
$0p_{1/2}$	2	[8]
$0p_{3/2}$	4	
$0s_{1/2}$	2	[2]



ASSIGNMENT OF J^π

Construct the **independent-particle wfn** for the ground state by filling up, in order of energy, the single-particle orbits



$$^{13}\text{C}: J^\pi = 1/2^-$$

A final question: what is the total ground state angular momentum and parity (π) of a given nuclide?

Any even number of particles (protons **or** neutrons) couple up to $J=0$. Therefore:

Any even-Z, even-N nuclides has $J^\pi = 0^+$

Any odd-A nucleus tends to have $J^\pi =$ that of **last** nucleon (Odd-Z, odd-N: no easy trend)

Next time: how to deal with excited states, odd-odd, etc.