



# HOW DO WE DO THIS?

Transition rate from  
Fermi's Golden Rule

$$R_{i \rightarrow f} = \frac{2\pi}{\hbar} \left| \langle \Psi_f | \hat{H}_{\text{int}} | \Psi_i \rangle \right|^2 \frac{dN}{dE}$$

transition matrix element

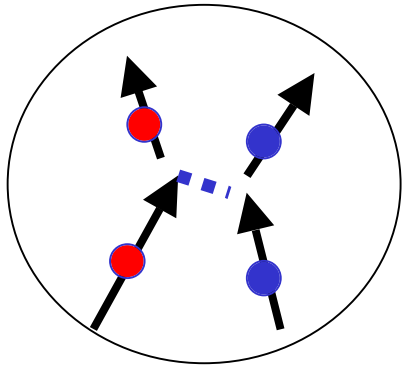
density of final states  
(phase space)

Cross-section  $\sigma = \frac{R_{i \rightarrow f}}{\text{flux}}$

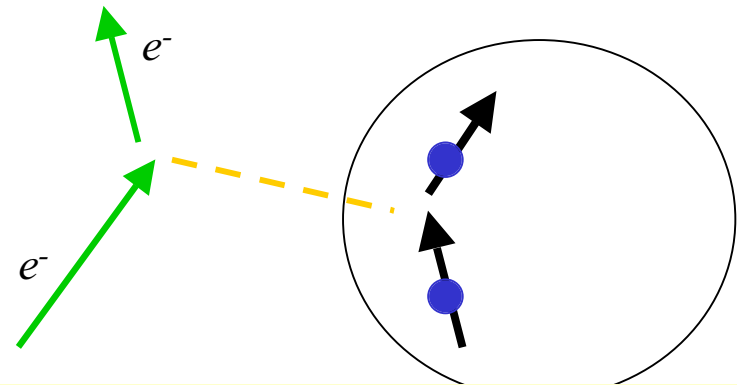
But what is  $H_{\text{int}}$ ? And how do we compute the transition matrix element?

Part A: interaction Hamiltonian  $\rightarrow$  *one body operator*  
Part B: matrix element of one-body operator between many-body states

## Part A: THE INTERACTION HAMILTONIAN



Inside the nucleus,  
**between** nucleons,  
a **two-body** operator  
 $= V(r_1, r_2)$

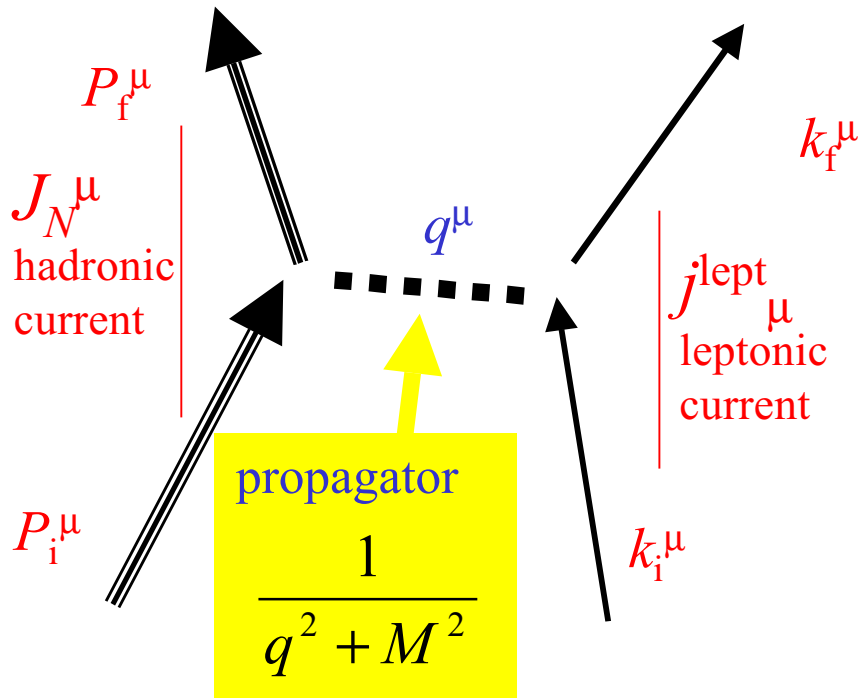


Interacting with the outside world  
(electrons, neutrinos, etc.)  
is a **one-body** operator **inside**  
the nucleus  $= \mathbf{O}(r)$

Steps for deriving the EM & weak transition operators:

1. Start with relativistic Dirac wavefunctions (**four-spinors**). Lorentz covariance restricts form of interactions.
2. Nucleon wavefunction: non-relativistic reduction of Dirac wavefunction.
3. **Leptons: plane waves, multipole expansion of  $e^{ikr}$**

# CURRENT-CURRENT HAMILTONIAN



4-momentum transfer  $q^\mu$   
 $= k_i^\mu - k_f^\mu = P_f^\mu - P_i^\mu$

$$\hat{H}_{\text{int}} = J_N^\mu \frac{1}{q^2 + M^2} j_\mu^{\text{lept}}$$

“Current-current interaction”

“vector”

“axial vector”

$$j_\mu = \bar{u}(p_f) \left\{ f_V(q^2) \gamma_\mu + f_A(q^2) \gamma_5 \gamma_\mu + \dots \right\} u(p_i)$$

Dirac spinor

Dirac matrix

form factor  
 (for nonpointlike  
 particles)

# MORE ON THE INTERACTION HAMILTONIAN

In propagator:  $\frac{1}{q^2 + M^2}$

$M =$  mass of exchanged boson  
 $= 0$  for photon (EM)  
 $\cong 90$  GeV for W, Z (weak)

Coupling “constants” in current:

**For leptons:**  $f_V = -e$  for electrons (EM)

weak interaction:  $f_V = -f_A = g = \text{“V-A”}$

**For nucleons:**  $f_V(q^2=0) = +e$  for protons (EM)

weak interaction:  $f_V(q^2=0) = g_V, f_A(q^2=0) = g_A$

$$g_A / g_V \approx 1.26$$

Non-relativistic  
 reduction of **nucleon**  
 wavefunction:

$$u_{nucl}(p) \approx \begin{pmatrix} 1 \\ \frac{\sigma \cdot p}{E + m_N} \end{pmatrix} \sqrt{\frac{m_N}{E + m_N}} \phi(\vec{r}) \chi$$

upper component  $\Rightarrow$  1

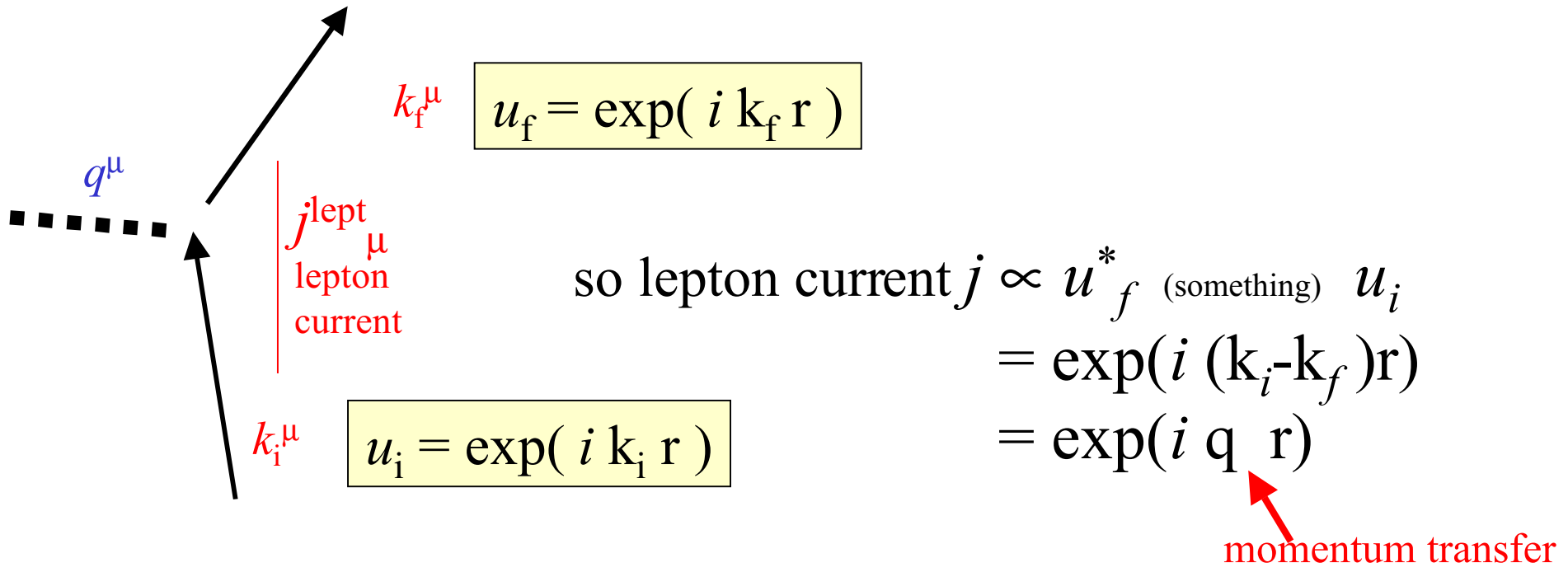
lower component  $\Rightarrow$   $\frac{\sigma \cdot p}{E + m_N}$

non-rel. spinor  $\Rightarrow$   $\chi$

spatial wfn  $\Rightarrow$   $\phi(\vec{r})$

Note: this is exact for plane waves

# EXPANSION OF LEPTON PLANE WAVES



**Key idea** multipole expansion of plane wave:

$$\exp(i \vec{q} \cdot \vec{r}) = \sum_{L=0} \sqrt{4\pi (2L+1)} i^L j_L(qr) Y_{L0}(\theta, \varphi)$$

spherical Bessel function  
(argument actually  $qr/\hbar$ )

spherical harmonic

# FORM OF ONE-BODY TRANSITION OPERATORS

We have **one-body** operators that acts on **nucleons**.

Examples:

$$j_L(qr) Y_{L0}(\theta, \varphi)$$

operates on nucleon spin

from multipole  
expansion of  
lepton current

$$j_L(qr) Y_{L0}(\theta, \varphi) \sigma$$

$\sigma$

from non-relativistic reduction  
of nucleon current (or from  
treatment of Dirac spinors in  
lepton wavefunction)

$$j_L(qr) Y_{L0}(\theta, \varphi) \nabla$$

$\nabla$

For experts: there are seven basic terms

density or charge

electric transverse

magnetic transverse

axial charge

axial electric transverse

axial magnetic transverse

axial longitudinal

from Lorentz-covariant currents

$\gamma_0$  = charge  $\gamma_{1,2}$  = E, M transverse

(no longitudinal  $\gamma_3$  because of

vector current conservation)

$\gamma_5 \gamma_0$  = axial charge

$\gamma_5 \gamma_{1,2}$  = axial E, M transverse

$\gamma_5 \gamma_3$  = axial longitudinal

## LONG-WAVELENGTH LIMIT

As with the Hamiltonian, we use these operators in integrals between single-particle wavefunctions:

$$\langle \phi_f | j_L Y_L | \phi_i \rangle = \int d^3 r \phi_f^*(\vec{r}) j_L(qr) Y_L(\theta, \varphi) \phi_i(\vec{r})$$

single-nucleon wfns (from HF, HO, WS...)

The single-nucleon wavefunctions have a finite extent, i.e., they die off for  $r > R = \text{nuclear radius}$ .

Typical  $R = \text{few fm}$ , but in **most** cases  $q = \text{a few MeV}$ ; then  $\lambda = \hbar c / q = 100 \text{ fm}$

**that is,  $qr < qR$  is always small**

Therefore the integral is governed by the behavior of  $j_L(qr)$  for  $r < R$

for small  $x$ ,  $j_L(x) \approx x^L / (2L+1)!!$   
that is, we want  $qR / \hbar c \ll 1$   
**this is the long-wavelength approximation**

## SAMPLE TRANSITION OPERATORS

$e^-$ -scatt. charge form factor:

$$j_0(qr)$$

E2 gamma decay:

$$r^2 Y_{20}$$

E2  $e^-$ -scatt. form factor

$$j_2(qr) Y_{20}$$

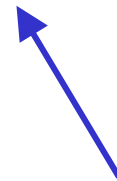
Fermi (beta) transition:

$$\tau_+$$

Gamow-Teller transition:

$$\sigma\tau_+$$

$\tau_+$  changes neutrons  
into protons



spin operator, acts on  
spin of neutron

## Part B: APPLICATION TO THE SHELL MODEL

Given a **one-body transition operator**  $\mathbf{O}(\mathbf{r})$ , how do we compute  $\langle \Psi_f | \mathbf{O} | \Psi_i \rangle$ ?

many-body wfns  
from shell-model calc.

Use the same approach as for the Hamiltonian:  
expand in creation/annihilation operators defined  
by single-particle basis:

$$\mathbf{O} = \langle i | \hat{\mathbf{O}} | j \rangle \hat{c}_i^* \hat{c}_j$$

$$\langle i | \hat{\mathbf{O}} | j \rangle = \int d^3r \phi_i^*(\vec{r}) \boxed{j_L(qr) Y_{L0}(\theta, \varphi)} \phi_j(\vec{r})$$

single-particle  
wfn

or some other  
one-body operator

Now we need to  
compute:

$$\langle \Psi_f | c_i^\dagger c_j | \Psi_i \rangle$$

many-body wfns

# THE (ONE-BODY) DENSITY MATRIX

We call  $\rho^{fi}(ab) = \langle \Psi_f | \hat{c}_a^* c_b | \Psi_i \rangle$   
the **one-body density matrix** also called the transition density matrix

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It can be computed exactly as the Hamiltonian matrix elements, by operating on the occupation representation of the basis states

example:

$$c_2^\dagger c_1 | 101011 \rangle = | 011011 \rangle$$

destroy particle in state #1

create particle in state #2

In slow motion:  $c_1 | 101011 \rangle = | 001011 \rangle$   
 $c_2^\dagger | 001011 \rangle = | 011011 \rangle$

## REDUCED MATRIX ELEMENTS

We always couple up to good angular momentum:

$$[c^\dagger \otimes c]_{JM} = \sum (j_1 m_1, j_2 m_2 | JM) (-1)^{j_2 - m_2} c^\dagger_{j_1, m_1} c_{j_2, -m_2}$$

Clebsch-Gordan coeff

Furthermore, we compute “reduced” matrix elements via Wigner-Eckhardt theorem (this removes dependence on orientation, or  $m$ 's:

$$\langle \Psi; J_f M_f | [c^* \quad c]_{\Delta J, \Delta M} | \Psi; J_i M_i \rangle =$$

$$(J_i M_i, \Delta J \Delta M | J_f M_f) \langle \Psi; J_f || [c^* \quad c]_{\Delta J} || \Psi; J_i \rangle$$

Clebsch-Gordan coeff reduced matrix element

So from the shell model wavefunctions

we compute a list of reduce one-body matrix elements

$$\rho_{\underline{J_f}, \underline{J_i}}(\underline{ab}; \underline{\Delta J}) = \langle \Psi; J_f || [a^* \quad b]_{\Delta J} || \Psi; J_i \rangle$$

final, initial J  
of many-body  
states

labels of  
single-particle  
states in transition

J of transition

# CALCULATION OF TRANSITION MATRIX ELEMENT

The final result:

$$\left\langle \Psi; J_f \left\| \hat{O}(\Delta J) \right\| \Psi; J_i \right\rangle = \sum_{ab} \rho_{J_f J_i}(ab; \Delta J) \left\langle a \left\| \hat{O}(\Delta J) \right\| b \right\rangle$$

transition matrix element for some one-body operator  $O$ 
density matrix elements from many-body wfns
one-body matrix elements

↑ sum over one-body orbits
↑

Note: these come from **integrals** and depend upon **assumption** of single-particle radial wavefunctions, i.e., HO, WS, HF etc..

Not very sensitive in long-wavelength approximation...but **is** for form factors and large momentum transfers

## SO WHAT CAN WE CALCULATE?

1. Gamma decay **rates/lifetimes** usually  $B(E2)$ -values etc. = “strength”
2. Electron scattering **form factors** must make assumption about mean-field potential (HO, WS) if at large momentum transfer
3. Beta decay, inverse beta decay, electron capture **rates/B-values**
4. Neutrino scattering **cross-section** same thing
5. WIMP scattering **cross-section** long-wavelength limit no longer valid
6. ....?

# NUCLEAR SHELL MODEL PROGRAMS

ANTOINE and NATHAN (Strasbourg) state-of-the-art, very large M-scheme basis sizes, **not publically available**.

OXBASH see [www.nscl.msu.edu/~brown/database.htm](http://www.nscl.msu.edu/~brown/database.htm)

Most widely-used shell-model program. Segregates M-scheme basis into bases with good J,T.

**in development: REDSTICK/ELDORADO [www.physics.sdsu.edu/eldorado](http://www.physics.sdsu.edu/eldorado)**  
**Intended to replace OXBASH but with capabilities similar to ANTOINE.**

Other shell model codes: DUSM, RITSSCHIL, Glasgow-Los Alamos, Oak Ridge-Rochester, MONSTER, VAMPIR, others in Scandanavia, Japan, ...